Integration By Parts
Part I - The Mathematics And A Simple Example

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January, 2020

Integration By Parts is a method of integration that is often useful when two functions are multiplied together. This method finds the integral of a product of functions in terms of the integral of the product of their derivative and antiderivative. It is frequently used to transform the antiderivative of a product of functions into an antiderivative for which a solution can be more easily found.

In this white paper we will develop the mathematics of integration by parts and use the mathematics to find the solution to the following hypothetical problem...

Our Hypothetical Problem

Assume that we have defined two functions $F(x)$ and $G(x)$. We want to solve the integral of the product of those two functions. The integral that we want to solve is...

$$I = \int\limits_a^b F(x)\,G(x)\,\delta x \;\text{ ...where... }\; F(x) = 2\,x^2 \;\text{ ...and... }\; G(x) = 4\,x^3 \;\text{ ...and... }\; a = 2 \;\text{ ...and... }\; b = 4 \tag{1}$$

Question 1: What is the solution to the integral above using standard integration techniques?

Question 2: What is the solution to the integral above using integration by parts?

The Mathematics

Using the product rule the equation for the derivative of the product of the two functions $f(u)$ and $g(u)$ is...

$$\frac{\delta\,f(u)\,g(u)}{\delta u} = f(u)\,\frac{\delta\,g(u)}{\delta u} + g(u)\,\frac{\delta\,f(u)}{\delta u} \tag{2}$$

Using Equation (2) above and integrating over the interval $[a, b]$ we get the following equation...

$$\int\limits_a^b \frac{\delta\,f(u)\,g(u)}{\delta u}\,\delta u = \int\limits_a^b f(u)\,\frac{\delta\,g(u)}{\delta u}\,\delta u + \int\limits_a^b g(u)\,\frac{\delta\,f(u)}{\delta u}\,\delta u \tag{3}$$

Noting that the integral of the derivative of a function is the original function evaluated at the bounds of integration we can rewrite Equation (3) above as...

$$f(b)\,g(b) - f(a)\,g(a) = \int\limits_a^b f(u)\,\frac{\delta\,g(u)}{\delta u}\,\delta u + \int\limits_a^b g(u)\,\frac{\delta\,f(u)}{\delta u}\,\delta u \tag{4}$$

Using Equation (4) above and rearranging terms we get...

$$\int\limits_a^b f(u)\,\frac{\delta\,g(u)}{\delta u}\,\delta u = f(b)\,g(b) - f(a)\,g(a) - \int\limits_a^b g(u)\,\frac{\delta\,f(u)}{\delta u}\,\delta u \tag{5}$$

The Point: If we can rewrite Equation (1) above in the form of Equation (5) above then we have an alternative (and maybe easier) way of solving the integral of the product of the two functions.

The Answer To Our Hypothetical Problem

Question 1: What is the solution to our problem using standard integration techniques?

$$I = \int_a^b F(x)\,G(x)\,\delta x = \int_a^b (2\,x^2 \times 4\,x^3)\,\delta x = \int_a^b 8\,x^5\,\delta x = \frac{8}{6}\,x^6 \Big[_2^4 = 5,376 \tag{6}$$

Question 2: What is the solution to our problem using integration by parts?

We will define two functions of the variable u as follows...

$$f(u) = 2\,u^2 \text{ ...such that... } \frac{\delta\,f(u)}{\delta u} = 4\,u \text{ ...and... } g(u) = u^4 \text{ ...such that... } \frac{\delta\,g(u)}{\delta u} = 4\,u^3 \tag{7}$$

Using the definitions in Equation (7) above we can rewrite Equation (1) above as...

$$I = \int_a^b F(x)\,G(x)\,\delta x = \int_a^b f(u)\,\frac{\delta\,g(u)}{\delta u}\,\delta u \text{ ...where... } F(x) = 2\,x^2 \text{ ...and... } G(x) = 4\,x^3 \tag{8}$$

Using Equation (5) above we can rewrite Equation (8) above as...

$$I = f(b)\,g(b) - f(a)\,g(a) - \int_a^b g(u)\,\frac{\delta\,f(u)}{\delta u}\,\delta u \tag{9}$$

The solution to the first half of Equation (9) above is...

$$f(b)\,g(b) - f(a)\,g(a) = 2\,b^2 \times b^4 - 2\,a^2 \times a^4 = 2\,b^6 - 2\,a^6 = 2 \times 4^6 - 2 \times 2^6 = 8,064 \tag{10}$$

The solution to the second half of Equation (9) above is...

$$\int_a^b g(u)\,\frac{\delta\,f(u)}{\delta u}\,\delta u = \int_a^b (u^4 \times 4\,u)\,\delta u = \int_a^b 4\,u^5\,\delta u = \frac{4}{6}\,u^6 \Big[_2^4 = 2,688 \tag{11}$$

Using Equations (9), (10) and (11) above the solution to our problem using integration by parts is...

$$I = 8,064 - 2,688 = 5,376 \tag{12}$$